18 Trigonometric Identities

INTRODUCTION

Recall that an algebraic equation is called an **algebraic identity** if it is true for all values of the variable(s) involved. Similarly, an equation involving trigonometric ratios of an angle is called a **trigonometric identity** if it is true for all values of the angle(s) involved.

In this chapter, we shall prove the basic trigonometric identities and use them further to prove some more trigonometric identities. We shall also simplify some algebraic trigonometric expressions by using the basic trigonometric identities. However, before doing so, let us review trigonometric ratios.

18.1 TRIGONOMETRIC RATIOS

Let ABC be a right angled triangle right angled at B *i.e.* $\angle B = 90°$. The side AC is its hypotenuse and $\angle A$ is an acute angle. Side BC is called ***side opposite to*** $\angle A$ (or *height*) and side AB is called ***side adjacent to*** $\angle A$ (or *base*).

The trigonometric ratios of acute $\angle A$ of a right angled triangle ABC right angled at B are defined as follows:

$$\text{sine } A = \frac{\text{side opposite to } \angle A}{\text{hypotenuse}} = \frac{BC}{AC}$$

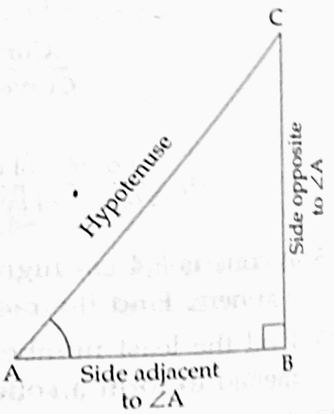
$$\text{cosine } A = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent } A = \frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \frac{BC}{AB}$$

$$\text{cotangent } A = \frac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A} = \frac{AB}{BC}$$

$$\text{secant } A = \frac{\text{hypotenuse}}{\text{side adjacent to } \angle A} = \frac{AC}{AB}$$

$$\text{cosecant } A = \frac{\text{hypotenuse}}{\text{side opposite to } \angle A} = \frac{AC}{BC}.$$



The trigonometric ratios defined above are abbreviated as sin A, cos A, tan A, cot A, sec A and cosec A respectively.

Thus, in a right angled triangle right angled at B *i.e.* $\angle B = 90°$, the side AC is its hypotenuse. In reference to acute $\angle A$, the side AB is called base and side BC is called height and the six trigonometric ratios (abbreviated *t*-ratios) are defined as follows:

(*i*) $\sin A = \dfrac{\text{height}}{\text{hypotenuse}} = \dfrac{BC}{AC}$

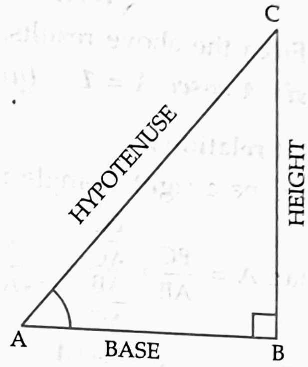
(*ii*) $\cos A = \dfrac{\text{base}}{\text{hypotenuse}} = \dfrac{AB}{AC}$

(*iii*) $\tan A = \dfrac{\text{height}}{\text{base}} = \dfrac{BC}{AB}$

(*iv*) $\cot A = \dfrac{\text{base}}{\text{height}} = \dfrac{AB}{BC}$

(*v*) $\sec A = \dfrac{\text{hypotenuse}}{\text{base}} = \dfrac{AC}{AB}$

(*vi*) $\operatorname{cosec} A = \dfrac{\text{hypotenuse}}{\text{height}} = \dfrac{AC}{BC}$.



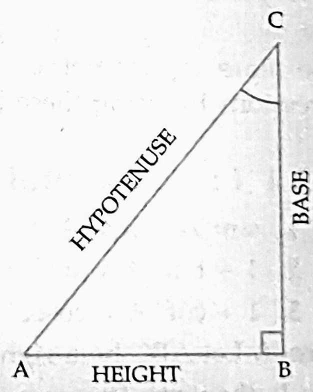
Remarks

1. Trigonometric ratios of an acute angle in a right angled triangle express the relationship between the angle and the lengths of its sides.
2. In a right angled triangle ABC, all trigonometric ratios of an acute angle A are positive real numbers.
3. For convenience, we write $\sin^2 A$, $\sin^3 A$ in place of $(\sin A)^2$, $(\sin A)^3$ etc., respectively.
4. In right triangle ABC right angled at B *i.e.* $\angle B = 90°$, if we consider $\angle C$ in place of $\angle A$ then the position of sides is changed (as shown in the adjoining figure). We have:

$$\sin C = \frac{AB}{AC},\ \cos C = \frac{BC}{AC},$$

$$\tan C = \frac{AB}{BC},\ \cot C = \frac{BC}{AB},$$

$$\sec C = \frac{AC}{BC},\ \operatorname{cosec} C = \frac{AC}{AB}.$$



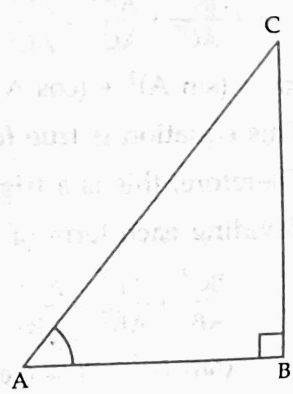
❐ Reciprocal relations

Let ABC be a right triangle right angled at B *i.e.* $\angle B = 90°$, then we have

(*i*) $\sin A = \dfrac{BC}{AC}$ and $\operatorname{cosec} A = \dfrac{AC}{BC}$

$\Rightarrow \quad \sin A = \dfrac{1}{\operatorname{cosec} A}$ and $\operatorname{cosec} A = \dfrac{1}{\sin A}$.

Thus, $\sin A = \dfrac{1}{\operatorname{cosec} A}$ *and* $\operatorname{cosec} A = \dfrac{1}{\sin A}$.



(*ii*) $\cos A = \dfrac{AB}{AC}$ and $\sec A = \dfrac{AC}{AB}$

$\Rightarrow \quad \cos A = \dfrac{1}{\sec A}$ and $\sec A = \dfrac{1}{\cos A}$.

Thus, $\cos A = \dfrac{1}{\sec A}$ *and* $\sec A = \dfrac{1}{\cos A}$.

(*iii*) $\tan A = \frac{BC}{AB}$ and $\cot A = \frac{AB}{BC}$

$\Rightarrow \quad \tan A = \frac{1}{\cot A}$ and $\cot A = \frac{1}{\tan A}$.

Thus, $\tan A = \frac{1}{\cot A}$ *and* $\cot A = \frac{1}{\tan A}$.

From the above results, it follows that

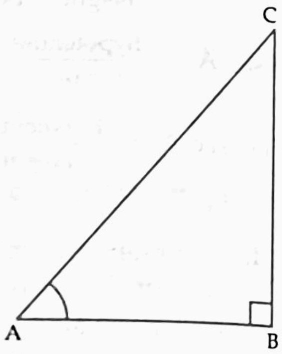
(*i*) $\sin A \operatorname{cosec} A = 1$ (*ii*) $\cos A \sec A = 1$ (*iii*) $\tan A \cot A = 1$.

❒ **Quotient relations**

Let ABC be a right triangle right angle at B *i.e.* ∠B = 90°, then we have:

(*i*) $\tan A = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin A}{\cos A}$

Thus, $\tan A = \frac{\sin A}{\cos A}$.



(*ii*) $\cot A = \frac{AB}{BC} = \frac{\frac{AB}{AC}}{\frac{BC}{AC}} = \frac{\cos A}{\sin A}$.

Thus, $\cot A = \frac{\cos A}{\sin A}$.

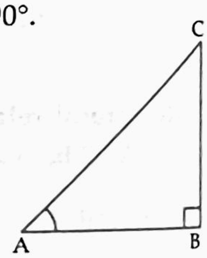
18.2 TRIGONOMETRIC IDENTITIES

Now, we shall prove the basic trigonometric identities and use them further to prove some more trigonometric identities. We shall also simplify some algebraic trigonometric expressions by using these basic trigonometric identities.

18.2.1 Fundamental identities

1. $\sin^2 A + \cos^2 A = 1$, $0° \le A \le 90°$
2. $1 + \tan^2 A = \sec^2 A$, $0° \le A < 90°$
3. $1 + \cot^2 A = \operatorname{cosec}^2 A$, $0° < A \le 90°$

Proof. Let ABC be a right angled triangle right angled at B *i.e.* ∠B = 90°.



By Pythagoras Theorem, we have

$BC^2 + AB^2 = AC^2$ ...(*i*)

Dividing each term of (*i*) by AC^2, we get

$\frac{BC^2}{AC^2} + \frac{AB^2}{AC^2} = \frac{AC^2}{AC^2}$ *i.e.* $\left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = 1$

$\Rightarrow \quad (\sin A)^2 + (\cos A)^2 = 1$ *i.e.* $\sin^2 A + \cos^2 A = 1$. ...(*ii*)

This equation is true for all values of A such that $0° \le A \le 90°$.

Therefore, this is a trigonometric identity.

Dividing each term of (*i*) by AB^2, we get

$\frac{BC^2}{AB^2} + \frac{AB^2}{AB^2} = \frac{AC^2}{AB^2}$ *i.e.* $\left(\frac{BC}{AB}\right)^2 + 1 = \left(\frac{AC}{AB}\right)^2$

$\Rightarrow \quad (\tan A)^2 + 1 = (\sec A)^2$ *i.e.* $1 + \tan^2 A = \sec^2 A$ ...(*iii*)

Note that tan A and sec A are not defined when A = 90°.

The equation (*iii*) is true for all values of A such that 0° ≤ A < 90°.

Therefore, equation (*iii*) is a trigonometric identity.

Dividing each term of (*i*) by BC^2, we get

$$\frac{BC^2}{BC^2} + \frac{AB^2}{BC^2} = \frac{AC^2}{BC^2} \quad i.e.\ 1 + \left(\frac{AB}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

$\Rightarrow$ $1 + (\cot A)^2 = (\text{cosec } A)^2$ *i.e.* $1 + \cot^2 A = \text{cosec}^2 A$. ...(*iv*)

Note that cot A and cosec A are not defined when A = 0°.

The equation (*iv*) is true for all values of A such that 0° < A ≤ 90°.

Therefore, equation (*iv*) is a trigonometric identity.

Thus, we have proved that:

$\sin^2 A + \cos^2 A = 1,\ 0° \le A \le 90°$ ...(*i*)

$1 + \tan^2 A = \sec^2 A,\ 0° \le A < 90°$ ...(*ii*)

$1 + \cot^2 A = \text{cosec}^2 A,\ 0° < A \le 90°$ ...(*iii*)

From the above results, it follows that:

(*i*) $1 - \sin^2 A = \cos^2 A,\ 1 - \cos^2 A = \sin^2 A$

(*ii*) $\sec^2 A - \tan^2 A = 1,\ \sec^2 A - 1 = \tan^2 A$

(*iii*) $\text{cosec}^2 A - \cot^2 A = 1,\ \text{cosec}^2 A - 1 = \cot^2 A$.

Remark

Using the above identities, we can express each trigonometric ratio in terms of other trigonometric ratios *i.e.* if the value of any one of the trigonometric ratios is known, then we can find the values of all other trigonometric ratios.

Illustrative Examples

Example 1. If A is an acute angle and $\tan A = \frac{12}{5}$, find all other trigonometric ratios of the angle A (using trigonometric identities).

Solution. Given $\tan A = \frac{12}{5}$

$\Rightarrow$ $\cot A = \frac{1}{\tan A} = \frac{5}{12}$;

$\sec^2 A = 1 + \tan^2 A = 1 + \left(\frac{12}{5}\right)^2 = \frac{25 + 144}{25} = \frac{169}{25}$

$\Rightarrow$ $\sec A = \frac{13}{5}$ (∵ sec A is +ve)

$\Rightarrow$ $\cos A = \frac{1}{\sec A} = \frac{5}{13}$;

$\sin A = \frac{\sin A}{\cos A} \cdot \cos A = \tan A \cos A = \frac{12}{5} \times \frac{5}{13} = \frac{12}{13}$

$\Rightarrow$ $\text{cosec } A = \frac{1}{\sin A} = \frac{13}{12}$.

Hence, $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\cot A = \frac{5}{12}$, $\sec A = \frac{13}{5}$, $\text{cosec } A = \frac{13}{12}$.

Example 2. Given A is an acute angle and 13 sin A = 5, evaluate $\frac{5\sin A - 2\cos A}{\tan A}$.

Solution. Given 13 sin A = 5 ⇒ $\sin A = \frac{5}{13}$.

We know that $\cos^2 A = 1 - \sin^2 A$

$$\Rightarrow \quad \cos^2 A = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{169-25}{169} = \frac{144}{169}$$

$$\Rightarrow \quad \cos A = \frac{12}{13}$$ (as A is an acute angle, cos A is +ve)

$$\tan A = \frac{\sin A}{\cos A} = \frac{5}{13} \times \frac{13}{12} = \frac{5}{12}.$$

$$\therefore \quad \frac{5\sin A - 2\cos A}{\tan A} = \frac{5 \times \frac{5}{13} - 2 \times \frac{12}{13}}{\frac{5}{12}} = \frac{\frac{1}{13}}{\frac{5}{12}} = \frac{1}{13} \times \frac{12}{5} = \frac{12}{65}.$$

Example 3. If $\tan\theta = \frac{1}{\sqrt{5}}$, find the value of $\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta}$.

Solution. Given $\tan\theta = \frac{1}{\sqrt{5}} \Rightarrow \cot\theta = \sqrt{5}$ $\left(\because \cot\theta = \frac{1}{\tan\theta}\right)$

Now $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{1}{\sqrt{5}}\right)^2 = 1 + \frac{1}{5} = \frac{6}{5}$

and $\text{cosec}^2\theta = 1 + \cot^2\theta = 1 + (\sqrt{5})^2 = 1 + 5 = 6.$

$$\therefore \quad \frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta} = \frac{6 - \frac{6}{5}}{6 + \frac{6}{5}} = \frac{30-6}{30+6} = \frac{24}{36} = \frac{2}{3}.$$

Example 4. Evaluate the following:

(i) $\frac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°}$ (ii) sin 25° cos 65° + cos 25° sin 65°.

Solution. (i) As sin 63° = sin (90° – 27°) = cos 27° and

cos 73° = cos (90° – 17°) = sin 17°,

$$\therefore \quad \frac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°} = \frac{\cos^2 27° + \sin^2 27°}{\cos^2 17° + \sin^2 17°} = \frac{1}{1} \quad (\because \sin^2 A + \cos^2 A = 1)$$

$= 1.$

(ii) sin 25° cos 65° + cos 25° sin 65°

= sin 25° cos (90° – 25°) + cos 25° sin (90° – 25°)

= sin 25° sin 25° + cos 25° cos 25°

$= \sin^2 25° + \cos^2 25°$

= 1 $(\because \sin^2 A + \cos^2 A = 1)$

Example 5. Without using trigonometrical tables, evaluate:

(i) $\sin^2 28° + \sin^2 62° + \tan^2 38° - \cot^2 52° + \frac{1}{4}\sec^2 30$ *(2017)*

(ii) $\text{cosec}^2 57° - \tan^2 33° + \cos 44° \text{ cosec } 46° - \sqrt{2}\cos 45° - \tan^2 60°$ *(2016)*

Solution. (i) As $\sin^2 62° = \sin^2(90° - 28°) = \cos^2 28°$ and

$\cot^2 52° = \cot^2(90° - 38°) = \tan^2 38°$,

$\therefore\ \sin^2 28^\circ + \sin^2 62^\circ + \tan^2 38^\circ - \cot^2 52^\circ + \frac{1}{4}\sec^2 30^\circ$

$= \sin^2 28^\circ + \cos^2 28^\circ + \tan^2 38^\circ - \tan^2 38^\circ + \frac{1}{4}\cdot\left(\frac{2}{\sqrt{3}}\right)^2$

$= 1 + \frac{1}{4}\times\frac{4}{3}$ $\quad (\because \sin^2 A + \cos^2 A = 1)$

$= 1 + \frac{1}{3} = \frac{4}{3}.$

(*ii*) As cosec 57° = cosec (90° – 33°) = sec 33° and cosec 46° = cosec (90° – 44°) = sec 44°

$\therefore\ \text{cosec}^2 57^\circ - \tan^2 33^\circ + \cos 44^\circ \text{ cosec } 46^\circ - \sqrt{2}\cos 45^\circ - \tan^2 60^\circ$

$= (\sec^2 33^\circ - \tan^2 33^\circ) + \cos 44^\circ \sec 44^\circ - \sqrt{2}\times\frac{1}{\sqrt{2}} - (\sqrt{3})^2$

$= 1 + 1 - 1 - 3$ $\quad (\because \sec^2 A - \tan^2 A = 1 \text{ and } \cos A \sec A = 1)$

$= -2.$

Example 6. Without using trigonometrical tables, evaluate:

(*i*) $\frac{2}{3}\text{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan 32^\circ - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$

(*ii*) $\left(\frac{\tan 20^\circ}{\text{cosec } 70^\circ}\right)^2 + \left(\frac{\cot 20^\circ}{\sec 70^\circ}\right)^2 + 2\tan 15^\circ \tan 45^\circ \tan 75^\circ.$

Solution. (*i*) $\frac{2}{3}\text{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan 32^\circ - \frac{5}{3}\tan 13^\circ \tan 37^\circ \tan 45^\circ \tan 53^\circ \tan 77^\circ$

$= \frac{2}{3}\text{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \tan (90^\circ - 58^\circ)$

$- \frac{5}{3}\tan 13^\circ \tan 37^\circ . 1 . \tan(90^\circ - 37^\circ) \tan (90^\circ - 13^\circ)$

$= \frac{2}{3}\text{cosec}^2 58^\circ - \frac{2}{3}\cot 58^\circ \cot 58^\circ - \frac{5}{3}\tan 13^\circ \tan 37^\circ \cot 37^\circ \cot 13^\circ$

$= \frac{2}{3}(\text{cosec}^2 58^\circ - \cot^2 58^\circ) - \frac{5}{3}\tan 13^\circ . 1 . \cot 13^\circ$

$= \frac{2}{3} . 1 - \frac{5}{3} . 1$ $\quad (\because \text{cosec}^2 A - \cot^2 A = 1)$

$= \frac{2}{3} - \frac{5}{3} = -1.$

(*ii*) $\left(\frac{\tan 20^\circ}{\text{cosec } 70^\circ}\right)^2 + \left(\frac{\cot 20^\circ}{\sec 70^\circ}\right)^2 + 2\tan 15^\circ \tan 45^\circ \tan 75^\circ$

$= \left(\frac{\tan 20^\circ}{\text{cosec } (90^\circ - 20^\circ)}\right)^2 + \left(\frac{\cot 20^\circ}{\sec (90^\circ - 20^\circ)}\right)^2 + 2\tan 15^\circ . 1 . \tan (90^\circ - 15^\circ)$

$= \left(\frac{\tan 20^\circ}{\sec 20^\circ}\right)^2 + \left(\frac{\cot 20^\circ}{\text{cosec } 20^\circ}\right)^2 + 2\tan 15^\circ \cot 15^\circ$

$= \left(\frac{\sin 20^\circ}{\cos 20^\circ} . \cos 20^\circ\right)^2 + \left(\frac{\cos 20^\circ}{\sin 20^\circ} . \sin 20^\circ\right)^2 + 2 . 1$

$= \sin^2 20^\circ + \cos^2 20^\circ + 2$

$= 1 + 2$ $\quad (\because \sin^2 A + \cos^2 A = 1)$

$= 3.$

Example 7. Prove the following identities:

(*i*) (cosec A + cot A) (1 – cos A) = sin A

(*ii*) sec A (1 – sin A) (sec A + tan A) = 1.

Solution. (*i*) L.H.S. = (cosec A + cot A) (1 – cos A)

$$= \left(\frac{1}{\sin A} + \frac{\cos A}{\sin A}\right)(1 - \cos A)$$

$$= \left(\frac{1+\cos A}{\sin A}\right)(1 - \cos A) = \frac{1-\cos^2 A}{\sin A}$$

$$= \frac{\sin^2 A}{\sin A} \qquad (\because 1 - \cos^2 A = \sin^2 A)$$

= sin A = R.H.S.

(*ii*) L.H.S. = sec A (1 – sin A) (sec A + tan A)

$$= \left(\frac{1}{\cos A}\right)(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1-\sin A)(1+\sin A)}{\cos^2 A} = \frac{1-\sin^2 A}{\cos^2 A} = \frac{\cos^2 A}{\cos^2 A} \qquad (\because 1 - \sin^2 A = \cos^2 A)$$

= 1 = R.H.S.

Example 8. Prove the following identities:

(*i*) $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$ (2017)

(*ii*) $\sec^2\theta - \frac{\sin^2\theta - 2\sin^4\theta}{2\cos^4\theta - \cos^2\theta} = 1.$

Solution. (*i*) L.H.S. $= \frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \frac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$

$$= \frac{\sin\theta(1-2(1-\cos^2\theta))}{\cos\theta(2\cos^2\theta-1)} = \frac{\sin\theta(2\cos^2\theta-1)}{\cos\theta(2\cos^2\theta-1)}$$

= tan θ = R.H.S.

(*ii*) L.H.S. $= \sec^2\theta - \frac{\sin^2\theta - 2\sin^4\theta}{2\cos^4\theta - \cos^2\theta} = \sec^2\theta - \frac{\sin^2\theta(1-2\sin^2\theta)}{\cos^2\theta(2\cos^2\theta-1)}$

$$= \sec^2\theta - \frac{\sin^2\theta(1-2(1-\cos^2\theta))}{\cos^2\theta(2\cos^2\theta-1)}$$

$$= \sec^2\theta - \frac{\sin^2\theta(2\cos^2\theta-1)}{\cos^2\theta(2\cos^2\theta-1)}$$

= (1 + tan² θ) – tan² θ = 1 = R.H.S.

Example 9. Prove the following identities:

(*i*) $\frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A} = \frac{2\sin A}{1 - 2\cos^2 A}$ (2002)

(*ii*) $(1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right) = \frac{1}{\sin^2 A - \sin^4 A}.$

Solution. (*i*) L.H.S. $= \frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A}$

$$= \frac{(\sin A - \cos A) + (\sin A + \cos A)}{\sin^2 A - \cos^2 A} = \frac{2\sin A}{(1-\cos^2 A) - \cos^2 A}$$

$$= \frac{2\sin A}{1 - 2\cos^2 A} = \text{R.H.S.}$$

(ii) L.H.S. $= (1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right)$

$= \sec^2 A + (1 + \cot^2 A) = \sec^2 A + \text{cosec}^2 A$

$= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$

$= \frac{1}{(1 - \sin^2 A)\sin^2 A} = \frac{1}{\sin^2 A - \sin^4 A}$.

Example 10. Prove the following identities:

(i) $\frac{\cos A}{1 + \sin A} + \tan A = \sec A$ (ii) $\sqrt{\frac{1 + \cos A}{1 - \cos A}} = \text{cosec } A + \cot A$.

Solution. (i) L.H.S. $= \frac{\cos A}{1 + \sin A} + \tan A$

$= \frac{\cos A}{1 + \sin A} \times \frac{1 - \sin A}{1 - \sin A} + \tan A$ (Note this step)

$= \frac{\cos A(1 - \sin A)}{1 - \sin^2 A} + \tan A = \frac{\cos A\,(1 - \sin A)}{\cos^2 A} + \tan A$

$= \frac{1 - \sin A}{\cos A} + \tan A = \frac{1}{\cos A} - \frac{\sin A}{\cos A} + \tan A$

$= \sec A - \tan A + \tan A = \sec A =$ R.H.S.

(ii) L.H.S. $= \sqrt{\frac{1 + \cos A}{1 - \cos A}} = \sqrt{\frac{1 + \cos A}{1 - \cos A} \times \frac{1 + \cos A}{1 + \cos A}}$

$= \sqrt{\frac{(1 + \cos A)^2}{1 - \cos^2 A}} = \sqrt{\frac{(1 + \cos A)^2}{\sin^2 A}}$

$= \frac{1 + \cos A}{\sin A} = \frac{1}{\sin A} + \frac{\cos A}{\sin A} = \text{cosec } A + \cot A =$ R.H.S.

Example 11. Prove the following identities:

(i) $\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \text{ cosec } A$ *(2009, 04)*

(ii) $\frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A} = \sin A + \cos A$ *(2015, 03)*

Solution. (i) L.H.S. $= \frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = \frac{\sin^2 A + (1 + \cos A)^2}{(1 + \cos A)\sin A}$

$= \frac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{(1 + \cos A)\sin A} = \frac{(\sin^2 A + \cos^2 A) + 1 + 2\cos A}{(1 + \cos A)\sin A}$

$= \frac{1 + 1 + 2\cos A}{(1 + \cos A)\sin A} = \frac{2(1 + \cos A)}{(1 + \cos A)\sin A}$

$= \frac{2}{\sin A} = 2 \text{ cosec } A =$ R.H.S.

(ii) L.H.S. $= \frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A}$

$= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}} = \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A}$

$$= \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A} = \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A}$$

$$= \frac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A - \sin A} = \cos A + \sin A$$

$$= \sin A + \cos A = \text{R.H.S.}$$

Example 12. Prove the following identities:

(*i*) $\tan^4 A + \tan^2 A = \sec^4 A - \sec^2 A$

(*ii*) $\dfrac{\cos^2 A + \tan^2 A - 1}{\sin^2 A} = \tan^2 A.$

Solution. (*i*) L.H.S. $= \tan^4 A + \tan^2 A$

$= \tan^2 A (\tan^2 A + 1)$

$= (\sec^2 A - 1) \sec^2 A$ $\quad (\because 1 + \tan^2 A = \sec^2 A)$

$= \sec^4 A - \sec^2 A = \text{R.H.S.}$

(*ii*) L.H.S. $= \dfrac{\cos^2 A + \tan^2 A - 1}{\sin^2 A} = \dfrac{\tan^2 A - (1 - \cos^2 A)}{\sin^2 A}$

$= \dfrac{\tan^2 A - \sin^2 A}{\sin^2 A} = \dfrac{\tan^2 A}{\sin^2 A} - \dfrac{\sin^2 A}{\sin^2 A}$

$= \dfrac{\sin^2 A}{\cos^2 A} \cdot \dfrac{1}{\sin^2 A} - 1 = \dfrac{1}{\cos^2 A} - 1$

$= \sec^2 A - 1 = \tan^2 A = \text{R.H.S.}$

Example 13. Prove the following identities:

(*i*) $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \text{cosec}\,\theta$ *(2014)*

(*ii*) $\dfrac{\sin\theta\tan\theta}{1 - \cos\theta} = 1 + \sec\theta$ *(2016)*

Solution. (*i*) L.H.S. $= (\sin\theta + \cos\theta)(\tan\theta + \cot\theta)$

$= (\sin\theta + \cos\theta)\left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)$

$= (\sin\theta + \cos\theta)\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= (\sin\theta + \cos\theta) \times \dfrac{1}{\cos\theta\sin\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta\sin\theta}$

$= \dfrac{\cos\theta}{\cos\theta\sin\theta} + \dfrac{\sin\theta}{\cos\theta\sin\theta} = \dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}$

$= \text{cosec}\,\theta + \sec\theta = \text{R.H.S.}$

(*ii*) L.H.S. $= \dfrac{\sin\theta\tan\theta}{1 - \cos\theta} = \dfrac{\sin\theta\tan\theta}{1 - \cos\theta} \times \dfrac{1 + \cos\theta}{1 + \cos\theta}$

$= \dfrac{\sin\theta\tan\theta(1 + \cos\theta)}{1 - \cos^2\theta} = \dfrac{\sin\theta\tan\theta(1 + \cos\theta)}{\sin^2\theta}$

$= \dfrac{\tan\theta(1 + \cos\theta)}{\sin\theta} = \dfrac{\dfrac{\sin\theta}{\cos\theta} \times (1 + \cos\theta)}{\sin\theta}$

$= \dfrac{1 + \cos\theta}{\cos\theta} = \dfrac{1}{\cos\theta} + \dfrac{\cos\theta}{\cos\theta}$

$= \sec\theta + 1 = 1 + \sec\theta = \text{R.H.S.}$

Example 14. Prove the following :

(*i*) (sec A – cos A) (tan A + cot A) = tan A sec A

(*ii*) (sin θ + cos θ + 1) (sin θ – 1 + cos θ) sec θ cosec θ = 2.

Solution. (*i*) L.H.S. = (sec A – cos A) (tan A + cot A)

$$= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$

$$= \frac{1-\cos^2 A}{\cos A} \cdot \frac{\sin^2 A + \cos^2 A}{\cos A \sin A} = \frac{\sin^2 A}{\cos A} \cdot \frac{1}{\cos A \sin A}$$

$$= \frac{\sin A}{\cos^2 A} = \frac{\sin A}{\cos A} \cdot \frac{1}{\cos A} = \tan A \sec A = \text{R.H.S.}$$

(*ii*) L.H.S. = (sin θ + cos θ + 1) (sin θ – 1 + cos θ) sec θ cosec θ

$$= (\overline{\sin\theta + \cos\theta} + 1)(\overline{\sin\theta + \cos\theta} - 1) \sec\theta \operatorname{cosec}\theta$$

$$= ((\sin\theta + \cos\theta)^2 - 1^2) \sec\theta \operatorname{cosec}\theta$$

$$= (\overline{\sin^2\theta + \cos^2\theta} + 2\sin\theta\cos\theta - 1) \sec\theta \operatorname{cosec}\theta$$

$$= (1 + 2\sin\theta\cos\theta - 1) \sec\theta \operatorname{cosec}\theta$$

$$= 2\sin\theta\cos\theta \times \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} = 2 = \text{R.H.S.}$$

Example 15. Prove the following identities:

(*i*) $(\operatorname{cosec}\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$

(*ii*) $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$.

Solution. (*i*) L.H.S. $= (\operatorname{cosec}\theta - \cot\theta)^2 = \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2 = \frac{(1-\cos\theta)^2}{\sin^2\theta}$

$$= \frac{(1-\cos\theta)^2}{1-\cos^2\theta} = \frac{(1-\cos\theta)^2}{(1+\cos\theta)(1-\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} = \text{R.H.S.}$$

(*ii*) L.H.S. $= (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$

$$= \sin^2 A + \operatorname{cosec}^2 A + 2\sin A \operatorname{cosec} A + \cos^2 A + \sec^2 A + 2\cos A \sec A$$

$$= (\sin^2 A + \cos^2 A) + (1 + \cot^2 A) + 2 \times 1 + (1 + \tan^2 A) + 2 \times 1$$

$$= 1 + 6 + \cot^2 A + \tan^2 A = 7 + \tan^2 A + \cot^2 A = \text{R.H.S.}$$

Example 16. Prove the following identities:

(*i*) sin θ (1 + tan θ) + cos θ (1 + cot θ) = sec θ + cosec θ

(*ii*) (1 + cot θ – cosec θ) (1 + tan θ + sec θ) = 2. *(2018)*

Solution. (*i*) L.H.S. = sin θ (1 + tan θ) + cos θ (1 + cot θ)

$$= \sin\theta\left(1 + \frac{\sin\theta}{\cos\theta}\right) + \cos\theta\left(1 + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin\theta\left(\frac{\cos\theta + \sin\theta}{\cos\theta}\right) + \cos\theta\left(\frac{\sin\theta + \cos\theta}{\sin\theta}\right)$$

$$= (\cos\theta + \sin\theta)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= (\cos\theta + \sin\theta)\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right) = (\cos\theta + \sin\theta)\left(\frac{1}{\cos\theta\sin\theta}\right)$$

$$= \frac{\cos\theta}{\cos\theta\sin\theta} + \frac{\sin\theta}{\cos\theta\sin\theta} = \frac{1}{\sin\theta} + \frac{1}{\cos\theta}$$

$= \text{cosec}\ \theta + \sec\theta = \text{R.H.S.}$

(*ii*) L.H.S. $= (1 + \cot\theta - \text{cosec}\ \theta)(1 + \tan\theta + \sec\theta)$

$$= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} = \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{R.H.S.}$$

Example 17. Prove the following identities:

(*i*) $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = 1 + \tan A + \cot A$

(*ii*) $\dfrac{\sin\theta}{\sec\theta + \tan\theta - 1} + \dfrac{\cos\theta}{\text{cosec}\ \theta + \cot\theta - 1} = 1.$

Solution. (*i*) L.H.S. $= \dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = \dfrac{\frac{\sin A}{\cos A}}{1 - \frac{\cos A}{\sin A}} + \dfrac{\frac{\cos A}{\sin A}}{1 - \frac{\sin A}{\cos A}}$

$$= \frac{\sin^2 A}{\cos A(\sin A - \cos A)} + \frac{\cos^2 A}{\sin A(\cos A - \sin A)}$$

$$= \frac{1}{\sin A - \cos A}\left(\frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A}\right)$$

$$= \frac{1}{\sin A - \cos A} \cdot \frac{\sin^3 A - \cos^3 A}{\cos A \sin A}$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \sin A\cos A + \cos^2 A)}{(\sin A - \cos A)\sin A\cos A}$$

$$= \frac{\sin^2 A + \sin A\cos A + \cos^2 A}{\sin A\cos A}$$

$$= \frac{\sin^2 A}{\sin A\cos A} + \frac{\sin A\cos A}{\sin A\cos A} + \frac{\cos^2 A}{\sin A\cos A}$$

$$= \frac{\sin A}{\cos A} + 1 + \frac{\cos A}{\sin A} = \tan A + 1 + \cot A$$

$= 1 + \tan A + \cot A = \text{R.H.S.}$

(*ii*) L.H.S. $= \dfrac{\sin\theta}{\sec\theta + \tan\theta - 1} + \dfrac{\cos\theta}{\text{cosec}\ \theta + \cot\theta - 1}$

$$= \frac{\sin\theta}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} - 1} + \frac{\cos\theta}{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} - 1}$$

$$= \frac{\sin\theta\cos\theta}{1+\sin\theta-\cos\theta} + \frac{\sin\theta\cos\theta}{1+\cos\theta-\sin\theta}$$

$$= \sin\theta\cos\theta\left(\frac{1}{1+(\sin\theta-\cos\theta)} + \frac{1}{1-(\sin\theta-\cos\theta)}\right)$$

$$= \sin\theta\cos\theta \cdot \frac{1-(\sin\theta-\cos\theta)+1+(\sin\theta-\cos\theta)}{1^2-(\sin\theta-\cos\theta)^2}$$

$$= \sin\theta\cos\theta \cdot \frac{2}{1-(\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta)}$$

$$= \frac{2\sin\theta\cos\theta}{1-(1-2\sin\theta\cos\theta)} = \frac{2\sin\theta\cos\theta}{2\sin\theta\cos\theta} = 1 = \text{R.H.S.}$$

Example 18. Prove the following identities:

(i) $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$

(ii) $\frac{\sec A + \tan A}{\text{cosec } A + \cot A} = \frac{\text{cosec } A - \cot A}{\sec A - \tan A}$

(iii) $(\text{cosec } \theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$ *(2019)*

(iv) $(1 + \cot A + \tan A)(\sin A - \cos A) = \frac{\sec A}{\text{cosec}^2 A} - \frac{\text{cosec } A}{\sec^2 A}$.

Solution. (i) L.H.S. $= \frac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$

$$= \frac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1} \quad (\because \sec^2 A - \tan^2 A = 1)$$

$$= \frac{(\tan A + \sec A) - (\sec A + \tan A)(\sec A - \tan A)}{\tan A - \sec A + 1}$$

$$= \frac{(\tan A + \sec A)(1 - \sec A + \tan A)}{\tan A - \sec A + 1}$$

$$= \tan A + \sec A = \frac{\sin A}{\cos A} + \frac{1}{\cos A}$$

$$= \frac{\sin A + 1}{\cos A} = \frac{1 + \sin A}{\cos A} = \text{R.H.S.}$$

(ii) L.H.S. $= \frac{\sec A + \tan A}{\text{cosec } A + \cot A}$

$$= \frac{\sec A + \tan A}{\text{cosec } A + \cot A} \times \frac{\text{cosec } A - \cot A}{\text{cosec } A - \cot A} \times \frac{\sec A - \tan A}{\sec A - \tan A}$$

$$= \frac{\text{cosec } A - \cot A}{\sec A - \tan A} \times \frac{(\sec A + \tan A)(\sec A - \tan A)}{(\text{cosec } A + \cot A)(\text{cosec } A - \cot A)}$$

$$= \frac{\text{cosec } A - \cot A}{\sec A - \tan A} \times \frac{\sec^2 A - \tan^2 A}{\text{cosec}^2 A - \cot^2 A}$$

$$= \frac{\text{cosec } A - \cot A}{\sec A - \tan A} \times \frac{1}{1} = \frac{\text{cosec } A - \cot A}{\sec A - \tan A} = \text{R.H.S.}$$

(iii) L.H.S. $= (\text{cosec } \theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^2\theta}{\cos\theta} \times \frac{1}{\cos\theta\sin\theta} = 1 = \text{R.H.S.}$$

(iv) L.H.S. = (1 + cot A + tan A) (sin A − cos A)

$$= \left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A)$$

$$= \left(\frac{\sin A \cos A + \cos^2 A + \sin^2 A}{\sin A \cos A}\right)(\sin A - \cos A)$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A \cos A)}{\sin A \cos A}$$

$$= \frac{\sin^3 A - \cos^3 A}{\sin A \cos A} \quad ...(1)$$

$$\text{R.H.S.} = \frac{\sec A}{\text{cosec}^2 A} - \frac{\text{cosec } A}{\sec^2 A} = \frac{\frac{1}{\cos A}}{\frac{1}{\sin^2 A}} - \frac{\frac{1}{\sin A}}{\frac{1}{\cos^2 A}}$$

$$= \frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A} = \frac{\sin^3 A - \cos^3 A}{\cos A \sin A} \quad ...(2)$$

From (1) and (2), L.H.S. = R.H.S.

Example 19. Prove the following identities:

(i) $\tan^2 \theta + \cot^2 \theta + 2 = \sec^2 \theta \text{ cosec}^2 \theta$

(ii) $\sqrt{\sec^2 \theta + \text{cosec}^2 \theta} = \tan \theta + \cot \theta.$ *(2018)*

Solution. (i) L.H.S. $= \tan^2 \theta + \cot^2 \theta + 2$

$$= (1 + \tan^2 \theta) + (1 + \cot^2 \theta)$$

$$= \sec^2 \theta + \text{cosec}^2 \theta$$

$$= \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}$$

$$= \frac{1}{\cos^2 \theta \sin^2 \theta} = \frac{1}{\cos^2 \theta} \cdot \frac{1}{\sin^2 \theta}$$

$$= \sec^2 \theta \text{ cosec}^2 \theta = \text{R.H.S.}$$

(ii) L.H.S. $= \sqrt{\sec^2 \theta + \text{cosec}^2 \theta}$

$$= \sqrt{(1 + \tan^2 \theta) + (1 + \cot^2 \theta)} = \sqrt{\tan^2 \theta + \cot^2 \theta + 2}$$

$$= \sqrt{\tan^2 \theta + \cot^2 \theta + 2 \tan \theta \cot \theta} \quad (\because \tan \theta \cot \theta = 1)$$

$$= \sqrt{(\tan \theta + \cot \theta)^2} = \tan \theta + \cot \theta = \text{R.H.S.}$$

Example 20. Prove the following identities:

(i) $\sin^4 \theta + \cos^4 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta$

(ii) $\frac{1}{\text{cosec}\,\theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\sin \theta} - \frac{1}{\text{cosec}\,\theta + \cot \theta}.$

Solution. (i) L.H.S. $= \sin^4 \theta + \cos^4 \theta = (\sin^2 \theta)^2 + (\cos^2 \theta)^2$

$$= (\sin^2 \theta + \cos^2 \theta)^2 - 2 \,.\, \sin^2 \theta \cos^2 \theta \quad (\because a^2 + b^2 = (a + b)^2 - 2ab)$$

$$= 1^2 - 2 \sin^2 \theta \cos^2 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = \frac{1}{\text{cosec}\,\theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\text{cosec}\,\theta - \cot \theta} \times \frac{\text{cosec}\,\theta + \cot \theta}{\text{cosec}\,\theta + \cot \theta} - \frac{1}{\sin \theta}$$

$$= \frac{\text{cosec}\,\theta + \cot \theta}{\text{cosec}^2 \theta - \cot^2 \theta} - \text{cosec}\,\theta$$

$$= \frac{\text{cosec}\,\theta + \cot\theta}{1} - \text{cosec}\,\theta \qquad (\because \text{cosec}^2\theta - \cot^2\theta = 1)$$

$$= \text{cosec}\,\theta + \cot\theta - \text{cosec}\,\theta = \cot\theta \qquad ...(1)$$

$$\text{R.H.S} = \frac{1}{\sin\theta} - \frac{1}{\text{cosec}\,\theta + \cot\theta} = \frac{1}{\sin\theta} - \frac{1}{\text{cosec}\,\theta + \cot\theta} \times \frac{\text{cosec}\,\theta - \cot\theta}{\text{cosec}\,\theta - \cot\theta}$$

$$= \text{cosec}\,\theta - \frac{\text{cosec}\,\theta - \cot\theta}{\text{cosec}^2\theta - \cot^2\theta} = \text{cosec}\,\theta - \frac{\text{cosec}\,\theta - \cot\theta}{1}$$

$$= \text{cosec}\,\theta - (\text{cosec}\,\theta - \cot\theta) = \cot\theta \qquad ...(2)$$

From (1) and (2), L.H.S = R.H.S.

Example 21. Prove the following identities:

(i) $\frac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta$

(ii) $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta = 1$

(iii) $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\,\text{cosec}\,\theta.$

Solution. (i)

$$\frac{1+\tan^2\theta}{1+\cot^2\theta} = \frac{\sec^2\theta}{\text{cosec}^2\theta} = \frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}$$

$$= \frac{1}{\cos^2\theta} \times \frac{\sin^2\theta}{1} = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta.$$

Also

$$\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \left(\frac{1-\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}}\right)^2 = \left(\frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}}\right)^2$$

$$= \left(-\frac{\sin\theta-\cos\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta-\cos\theta}\right)^2 = \left(-\frac{\sin\theta}{\cos\theta}\right)^2$$

$$= (-\tan\theta)^2 = \tan^2\theta.$$

$$\therefore \quad \frac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta.$$

(ii) L.H.S. $= \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta)^3 + (\cos^2\theta)^3 + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta) + 3\sin^2\theta\cos^2\theta$

$(\because a^3 + b^3 = (a+b)^3 - 3ab(a+b))$

$= (1)^3 - 3\sin^2\theta\cos^2\theta(1) + 3\sin^2\theta\cos^2\theta$

$= 1 - 3\sin^2\theta\cos^2\theta + 3\sin^2\theta\cos^2\theta = 1 =$ R.H.S.

(iii) L.H.S. $= \frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}}$

$$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} + \frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}$$

$$= \frac{1}{\sin\theta-\cos\theta}\left(\frac{\sin^2\theta}{\cos\theta} - \frac{\cos^2\theta}{\sin\theta}\right) = \frac{1}{\sin\theta-\cos\theta} \times \frac{\sin^3\theta-\cos^3\theta}{\sin\theta\cos\theta}$$

$$= \frac{(\sin\theta - \cos\theta)(\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta)}{(\sin\theta - \cos\theta)\sin\theta\cos\theta} \quad (\because a^3 - b^3 = (a-b)(a^2 + b^2 + ab))$$

$$= \frac{1 + \sin\theta\cos\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} + 1$$

$$= \sec\theta \operatorname{cosec}\theta + 1 = \text{R.H.S.}$$

Example 22. Given that $\alpha + \beta = 90°$, show that $\sqrt{\cos\alpha \operatorname{cosec}\beta - \cos\alpha \sin\beta} = \sin\alpha$.

Solution. Given $\alpha + \beta = 90° \Rightarrow \beta = 90° - \alpha$. ...(*i*)

$$\sqrt{\cos\alpha\operatorname{cosec}\beta - \cos\alpha\sin\beta} = \sqrt{\cos\alpha\operatorname{cosec}(90° - \alpha) - \cos\alpha\sin(90° - \alpha)} \quad \text{(using (i))}$$

$$= \sqrt{\cos\alpha\sec\alpha - \cos\alpha\cos\alpha} = \sqrt{1 - \cos^2\alpha} \quad (\because \cos\alpha\sec\alpha = 1)$$

$$= \sqrt{\sin^2\alpha} = \sin\alpha.$$

Example 23. If $2\sin^2\theta - \cos^2\theta = 2$, then find the value of θ.

Solution. Given $2\sin^2\theta - \cos^2\theta = 2$

$\Rightarrow \quad 2\sin^2\theta - (1 - \sin^2\theta) = 2$

$\Rightarrow \quad 3\sin^2\theta - 1 = 2 \Rightarrow 3\sin^2\theta = 3$

$\Rightarrow \quad \sin^2\theta = 1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = 90°$.

Hence, the value of θ is $90°$.

Example 24. If $1 + \sin^2\theta = 3\sin\theta\cos\theta$, prove that $\tan\theta = 1$ or $\frac{1}{2}$.

Solution. Given $1 + \sin^2\theta = 3\sin\theta\cos\theta$.

Dividing both sides by $\cos^2\theta$, we get

$$\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = 3\frac{\sin\theta\cos\theta}{\cos^2\theta}$$

$\Rightarrow \quad \sec^2\theta + \tan^2\theta = 3\tan\theta \Rightarrow (1 + \tan^2\theta) + \tan^2\theta = 3\tan\theta$

$\Rightarrow \quad 2\tan^2\theta - 3\tan\theta + 1 = 0 \Rightarrow 2\tan^2\theta - 2\tan\theta - \tan\theta + 1 = 0$

$\Rightarrow \quad 2\tan\theta(\tan\theta - 1) - 1(\tan\theta - 1) = 0$

$\Rightarrow \quad (\tan\theta - 1)(2\tan\theta - 1) = 0 \Rightarrow \tan\theta - 1 = 0$ or $2\tan\theta - 1 = 0$

$\Rightarrow \quad \tan\theta = 1$ or $\tan\theta = \frac{1}{2}$.

Example 25. If $a\sin\theta + b\cos\theta = c$, then prove that

$$a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}.$$

Solution. Given $a\sin\theta + b\cos\theta = c$

$\Rightarrow \quad (a\sin\theta + b\cos\theta)^2 = c^2$ (squaring both sides)

$\Rightarrow \quad a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2(1 - \cos^2\theta) + b^2(1 - \sin^2\theta) + 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2 + b^2 - (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta) = c^2$

$\Rightarrow \quad a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = a^2 + b^2 - c^2$

$\Rightarrow \quad (a\cos\theta - b\sin\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow \quad a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

Example 26. (*i*) If $\sin\theta + \sin^2\theta = 1$, prove that $\cos^2\theta + \cos^4\theta = 1$.

(*ii*) If $\tan^4\theta + \tan^2\theta = 1$, prove that $\cos^4\theta + \cos^2\theta = 1$.

Solution. (*i*) Given $\sin\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \sin\theta = \cos^2\theta \Rightarrow \sin^2\theta = \cos^4\theta$

$\Rightarrow 1 - \cos^2\theta = \cos^4\theta \Rightarrow \cos^2\theta + \cos^4\theta = 1.$

(*ii*) Given $\tan^4\theta + \tan^2\theta = 1 \Rightarrow \tan^2\theta(\tan^2\theta + 1) = 1$

$\Rightarrow 1 + \tan^2\theta = \frac{1}{\tan^2\theta} \Rightarrow \sec^2\theta = \cot^2\theta$

$\Rightarrow \frac{1}{\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} \Rightarrow \sin^2\theta = \cos^4\theta$

$\Rightarrow 1 - \cos^2\theta = \cos^4\theta \Rightarrow \cos^4\theta + \cos^2\theta = 1.$

Example 27. (*i*) If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, show that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

(*ii*) If $\sin\theta + 2\cos\theta = 1$, prove that $\cos\theta - 2\sin\theta = 2$.

Solution. (*i*) Given $\cos\theta + \sin\theta = \sqrt{2}\cos\theta \Rightarrow \sin\theta = (\sqrt{2} - 1)\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}-1}\sin\theta = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\sin\theta = \frac{(\sqrt{2}+1)\sin\theta}{2-1}$

$\Rightarrow \cos\theta = \sqrt{2}\sin\theta + \sin\theta \Rightarrow \cos\theta - \sin\theta = \sqrt{2}\sin\theta.$

(*ii*) Given $\sin\theta + 2\cos\theta = 1 \Rightarrow 2\cos\theta = 1 - \sin\theta$

$\Rightarrow 2\cos\theta(1 + \sin\theta) = (1 - \sin\theta)(1 + \sin\theta)$

$\Rightarrow 2\cos\theta(1 + \sin\theta) = 1 - \sin^2\theta$

$\Rightarrow 2\cos\theta(1 + \sin\theta) = \cos^2\theta$

$\Rightarrow 2(1 + \sin\theta) = \cos\theta$

$\Rightarrow \cos\theta - 2\sin\theta = 2.$

Example 28. If $\tan\theta + \sec\theta = l$, then prove that $\sec\theta = \frac{l^2+1}{2l}$.

Solution. Given $\tan\theta + \sec\theta = l$ ...(1)

We know that $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$\Rightarrow l(\sec\theta - \tan\theta) = 1$ (using (1))

$\Rightarrow \sec\theta - \tan\theta = \frac{1}{l}$ ...(2)

On adding (1) and (2), we get

$2\sec\theta = l + \frac{1}{l} \Rightarrow \sec\theta = \frac{l^2+1}{2l}.$

Example 29. (*i*) If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, prove that $x^2 + y^2 = a^2 + b^2$.

(*ii*) If $x = a\,\mathrm{cosec}\,\theta + b\cot\theta$ and $y = a\cot\theta + b\,\mathrm{cosec}\,\theta = y$, prove that $x^2 - y^2 = a^2 - b^2$.

Solution. (*i*) Given $x = a\cos\theta + b\sin\theta$ ...(1)

and $y = a\sin\theta - b\cos\theta$ ...(2)

On squaring (1) and (2) and then adding, we get

$$x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$
$$= a^2 \cdot 1 + b^2 \cdot 1 = a^2 + b^2.$$

(ii) Given $x = a \operatorname{cosec} \theta + b \cot \theta$ ...(1)

and $y = a \cot \theta + b \operatorname{cosec} \theta$ ...(2)

On squaring (1) and (2) and then subtracting, we get

$$x^2 - y^2 = a^2(\operatorname{cosec}^2 \theta - \cot^2 \theta) + b^2(\cot^2 \theta - \operatorname{cosec}^2 \theta)$$
$$= a^2 . 1 + b^2 (-1) = a^2 - b^2.$$

Example 30. If $\sin \theta + \cos \theta = p$ and $\sec \theta + \operatorname{cosec} \theta = q$, then prove that $q (p^2 - 1) = 2p$.

Solution. Given $\sin \theta + \cos \theta = p$ ...(i) and $\sec \theta + \operatorname{cosec} \theta = q$ ...(ii)

$$\Rightarrow \quad \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = q \Rightarrow \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = q$$

$$\Rightarrow \quad \frac{p}{\sin\theta\cos\theta} = q \quad \text{(using (i))}$$

$$\Rightarrow \quad \sin \theta \cos \theta = \frac{p}{q} \quad \text{...(iii)}$$

On squaring (i), we get

$$(\sin \theta + \cos \theta)^2 = p^2 = (\sin^2 \theta + \cos^2 \theta) + 2 \sin \theta \cos \theta = p^2$$

$$\Rightarrow \quad 1 + 2\frac{p}{q} = p^2 \quad \text{(using (iii))}$$

$$\Rightarrow \quad \frac{2p}{q} = p^2 - 1 \Rightarrow 2p = q(p^2 - 1).$$

Example 31. If $\tan \theta + \sin \theta = m$ and $\tan \theta - \sin \theta = n$, then prove that $m^2 - n^2 = 4\sqrt{mn}$.

Solution. Given $\tan \theta + \sin \theta = m$ and $\tan \theta - \sin \theta = n$

$$\Rightarrow \quad \tan \theta = \frac{m+n}{2} \text{ and } \sin \theta = \frac{m-n}{2}$$

$$\Rightarrow \quad \cot \theta = \frac{2}{m+n} \text{ and } \operatorname{cosec} \theta = \frac{2}{m-n}.$$

Now using $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$, we get

$$\left(\frac{2}{m-n}\right)^2 - \left(\frac{2}{m+n}\right)^2 = 1$$

$$\Rightarrow \quad 4[(m+n)^2 - (m-n)^2] = (m-n)^2 (m+n)^2$$

$$\Rightarrow \quad 4 \times 4\, mn = (m^2 - n^2)^2 \Rightarrow m^2 - n^2 = 4\sqrt{mn}.$$

Exercise 18

1. If A is an acute angle and $\sin A = \frac{3}{5}$, find all other trigonometric ratios of angle A (using trigonometric identities).
2. If A is an acute angle and $\sec A = \frac{17}{8}$, find all other trigonometric ratios of angle A (using trigonometric identities).
3. If $12 \operatorname{cosec} \theta = 13$, find the value of $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta}$.

Hint

$\cot \theta = \sqrt{\operatorname{cosec}^2 \theta - 1} = \sqrt{\left(\frac{13}{12}\right)^2 - 1} = \frac{5}{12}$.

Without using trigonometric tables, evaluate the following (4 to 7):

(*i*) $\cos^2 26° + \cos 64° \sin 26° + \dfrac{\tan 36°}{\cot 54°}$ (*2012*)

(*ii*) $\dfrac{\sec 17°}{\operatorname{cosec} 73°} + \dfrac{\tan 68°}{\cot 22°} + \cos^2 44° + \cos^2 46°$ (*2009*)

(*i*) $\dfrac{\sin 65°}{\cos 25°} + \dfrac{\cos 32°}{\sin 58°} - \sin 28° \sec 62° + \operatorname{cosec}^2 30°$ (*2015*)

(*ii*) $\dfrac{\sec 29°}{\operatorname{cosec} 61°} + 2 \cot 8° \cot 17° \cot 45° \cot 73° \cot 82° - 3(\sin^2 38° + \sin^2 52°)$.

(*i*) $\dfrac{\sin 35° \cos 55° + \cos 35° \sin 55°}{\operatorname{cosec}^2 10° - \tan^2 80°}$ (*2010*)

(*ii*) $\sin^2 34° + \sin^2 56° + 2 \tan 18° \tan 72° - \cot^2 30°$ (*2014*)

(*i*) $\left(\dfrac{\tan 25°}{\operatorname{cosec} 65°}\right)^2 + \left(\dfrac{\cot 25°}{\sec 65°}\right)^2 + 2 \tan 18° \tan 45° \tan 72°$

(*ii*) $(\cos^2 25° + \cos^2 65°) + \operatorname{cosec} \theta \sec (90° - \theta) - \cot \theta \tan (90° - \theta)$.

Prove the following (8 to 26) identities, where the angles involved are acute angles for which the trigonometric ratios as defined:

(*i*) $(\sec A + \tan A)(1 - \sin A) = \cos A$

(*ii*) $(1 + \tan^2 A)(1 - \sin A)(1 + \sin A) = 1$.

(*i*) $\tan A + \cot A = \sec A \operatorname{cosec} A$

(*ii*) $(1 - \cos A)(1 + \sec A) = \tan A \sin A$.

(*i*) $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$ (*ii*) $1 + \dfrac{\tan^2 A}{1 + \sec A} = \sec A$

(*iii*) $\dfrac{1 + \sec A}{\sec A} = \dfrac{\sin^2 A}{1 - \cos A}$ (*iv*) $\dfrac{\sin A}{1 - \cos A} = \operatorname{cosec} A + \cot A$.

(*i*) $\dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}$ (*ii*) $\dfrac{1 - \tan^2 A}{\cot^2 A - 1} = \tan^2 A$

(*iii*) $\dfrac{\sin A}{1 + \cos A} = \operatorname{cosec} A - \cot A$ (*2008*)

(*i*) $\dfrac{\sec A - 1}{\sec A + 1} = \dfrac{1 - \cos A}{1 + \cos A}$ (*2007*) (*ii*) $\dfrac{\tan^2 \theta}{(\sec \theta - 1)^2} = \dfrac{1 + \cos \theta}{1 - \cos \theta}$ (*2012*)

(*iii*) $(1 + \tan A)^2 + (1 - \tan A)^2 = 2 \sec^2 A$ (*2005*)

(*iv*) $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A$.

(*i*) $\dfrac{1 + \sin A}{\cos A} + \dfrac{\cos A}{1 + \sin A} = 2 \sec A$ (*ii*) $\dfrac{\tan A}{\sec A - 1} + \dfrac{\tan A}{\sec A + 1} = 2 \operatorname{cosec} A$.

(*i*) $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2 \sec^2 A$ (*ii*) $\cot A - \tan A = \dfrac{2\cos^2 A - 1}{\sin A \cos A}$

(*iii*) $\dfrac{\cot A - 1}{2 - \sec^2 A} = \dfrac{\cot A}{1 + \tan A}$.

(*i*) $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ (*ii*) $\dfrac{\cos \theta}{1 - \tan \theta} - \dfrac{\sin^2 \theta}{\cos \theta - \sin \theta} = \cos \theta + \sin \theta$.

16. (i) $\text{cosec}^4 \theta - \text{cosec}^2 \theta = \cot^4 \theta + \cot^2 \theta$

(ii) $2 \sec^2 \theta - \sec^4 \theta - 2 \text{cosec}^2 \theta + \text{cosec}^4 \theta = \cot^4 \theta - \tan^4 \theta$.

17. (i) $\frac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot \theta$ (ii) $\frac{\tan^3\theta-1}{\tan\theta-1} = \sec^2 \theta + \tan \theta$.

18. (i) $\frac{1+\text{cosec } A}{\text{cosec } A} = \frac{\cos^2 A}{1-\sin A}$ (ii) $\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A}$

19. (i) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \tan A + \sec A$ (ii) $\sqrt{\frac{1-\cos A}{1+\cos A}} = \text{cosec } A - \cot A$. (2013)

20. (i) $\sqrt{\frac{\sec A-1}{\sec A+1}} + \sqrt{\frac{\sec A+1}{\sec A-1}} = 2 \text{ cosec } A$ (ii) $\frac{\cos A \cot A}{1-\sin A} = 1 + \text{cosec } A$.

21. (i) $\frac{1+\tan A}{\sin A} + \frac{1+\cot A}{\cos A} = 2 (\sec A + \text{cosec } A)$

(ii) $\sec^4 A - \tan^4 A = 1 + 2 \tan^2 A$.

22. (i) $\text{cosec}^6 A - \cot^6 A = 3 \cot^2 A \text{ cosec}^2 A + 1$

(ii) $\sec^6 A - \tan^6 A = 1 + 3 \tan^2 A + 3 \tan^4 A$.

23. (i) $\frac{\cot\theta+\text{cosec}\,\theta-1}{\cot\theta-\text{cosec}\,\theta+1} = \frac{1+\cos\theta}{\sin\theta}$ (ii) $\frac{\sin\theta}{\cot\theta+\text{cosec}\,\theta} = 2 + \frac{\sin\theta}{\cot\theta-\text{cosec}\,\theta}$.

24. (i) $(\sin \theta + \cos \theta) (\sec \theta + \text{cosec } \theta) = 2 + \sec \theta \text{ cosec } \theta$

(ii) $(\text{cosec } A - \sin A) (\sec A - \cos A) \sec^2 A = \tan A$. (2011)

25. (i) $\frac{\sin^3 A+\cos^3 A}{\sin A+\cos A} + \frac{\sin^3 A-\cos^3 A}{\sin A-\cos A} = 2$

(ii) $\frac{\tan^2 A}{1+\tan^2 A} + \frac{\cot^2 A}{1+\cot^2 A} = 1$.

26. (i) $\frac{1}{\sec A+\tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A-\tan A}$

(ii) $(\sin A + \sec A)^2 + (\cos A + \text{cosec } A)^2 = (1 + \sec A \text{ cosec } A)^2$

(iii) $\frac{\tan A+\sin A}{\tan A-\sin A} = \frac{\sec A+1}{\sec A-1}$.

27. If $\sin \theta + \cos \theta = \sqrt{2} \sin (90° - \theta)$, show that $\cot \theta = \sqrt{2} + 1$.

28. If $7 \sin^2 \theta + 3 \cos^2 \theta = 4$, $0° \leq \theta \leq 90°$, then find the value of θ.

29. If $\sec \theta + \tan \theta = m$ and $\sec \theta - \tan \theta = n$, prove that $mn = 1$.

30. If $x = a \sec \theta + b \tan \theta$ and $y = a \tan \theta + b \sec \theta$, prove that $x^2 - y^2 = a^2 - b^2$.

31. If $x = h + a \cos \theta$ and $y = k + a \sin \theta$, prove that $(x - h)^2 + (y - k)^2 = a^2$.

Multiple Choice Questions

Choose the correct answer from the given four options (1 to 10):

1. $\cot^2 \theta - \frac{1}{\sin^2\theta}$ is equal to

(a) 1 (b) −1 (c) $\sin^2 \theta$ (d) $\sec^2 \theta$

2. $(\sec^2 \theta - 1) (1 - \text{cosec}^2 \theta)$ is equal to

(a) −1 (b) 1 (c) 0 (d) 2

3. $\frac{\tan^2 \theta}{1 + \tan^2 \theta}$ is equal to
   (*a*) $2 \sin^2 \theta$ (*b*) $2 \cos^2 \theta$ (*c*) $\sin^2 \theta$ (*d*) $\cos^2 \theta$
4. $(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2$ is equal to
   (*a*) -2 (*b*) 0 (*c*) 1 (*d*) 2
5. $(\sec A + \tan A)(1 - \sin A)$ is equal to
   (*a*) $\sec A$ (*b*) $\sin A$ (*c*) $\text{cosec } A$ (*d*) $\cos A$
6. $\frac{1 + \tan^2 A}{1 + \cot^2 A}$ is equal to
   (*a*) $\sec^2 A$ (*b*) -1 (*c*) $\cot^2 A$ (*d*) $\tan^2 A$
7. If $\sec \theta - \tan \theta = k$, then the value of $\sec \theta + \tan \theta$ is
   (*a*) $1 - \frac{1}{k}$ (*b*) $1 - k$ (*c*) $1 + k$ (*d*) $\frac{1}{k}$
8. If θ is an acute angle of a right triangle, then the value of $\sin \theta \cos (90° - \theta) + \cos \theta \sin (90° - \theta)$ is
   (*a*) 0 (*b*) $2 \sin \theta \cos \theta$ (*c*) 1 (*d*) $2 \sin^2 \theta$
9. The value of $\cos 65° \sin 25° + \sin 65° \cos 25°$ is
   (*a*) 0 (*b*) 1 (*c*) 2 (*d*) 4
10. The value of $3 \tan^2 26° - 3 \text{cosec}^2 64°$ is
    (*a*) 0 (*b*) 3 (*c*) -3 (*d*) -1

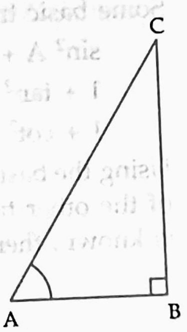
Summary

- **Trigonometric ratios**

  In a right triangle ABC, right angled at B,

  $\sin A = \frac{BC}{AC}$, $\cos A = \frac{AB}{AC}$, $\tan A = \frac{BC}{AB}$,

  $\cot A = \frac{AB}{BC}$, $\sec A = \frac{AC}{AB}$, $\text{cosec } A = \frac{AC}{BC}$.



- **Reciprocal relations**

  $\text{cosec } A = \frac{1}{\sin A}$, $\sec A = \frac{1}{\cos A}$, $\cot A = \frac{1}{\tan A}$.

- **Quotient relations**

  $\tan A = \frac{\sin A}{\cos A}$, $\cot A = \frac{\cos A}{\sin A}$.

- The trigonometric ratios of an acute angle in a right triangle express the relationship between the angle and the sides of the triangle.

- **Trigonometric ratios of some specific angles**

  Trigonometric ratios of 0°, 30°, 45°, 60° and 90° are given in the following table for ready reference:

∠A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

- In a right angled triangle ABC, all trigonometric ratios of an acute angle are positive real numbers. The values of sin A or cos A never exceed 1, whereas the values of sec A and cosec A are always greater than or equal to 1; tan A and cot A can take any positive real value.
- As A increases from 0° to 90°, the value of sin A increases whereas the value of cos A decreases.
- tan A and sec A are not defined when A = 90°; cot A and cosec A are not defined when A = 0°.

Trigonometric ratios of complementary angles

sin (90° – A) = cos A, cos (90° – A) = sin A,

tan (90° – A) = cot A, cot (90° – A) = tan A,

sec (90° – A) = cosec A, cosec (90°– A) = sec A.

Trigonometric identities

- An equation involving trigonometric ratios of an angle(s) is called a trigonometric identity if it is true for all values of the angle(s) involved.
- Some basic trigonometric identities are

$\sin^2 A + \cos^2 A = 1$, $0° \le A \le 90°$;

$1 + \tan^2 A = \sec^2 A$, $0° \le A < 90°$ and

$1 + \cot^2 A = \text{cosec}^2 A$, $0° < A \le 90°$

- Using the basic trigonometric identities, we can express each trigonometric ratio in terms of the other trigonometric ratios *i.e.* if the value of any one of the trigonometric ratios is known, then we can find the values of all other trigonometric ratios.

Chapter Test

1. (*i*) If θ is an acute angle and cosec θ = $\sqrt{5}$, find the value of cot θ – cos θ.

   (*ii*) If θ is an acute angle and tan θ = $\frac{8}{15}$, find the value of sec θ + cosec θ.

2. Evaluate the following:

   (*i*) $2 \times \left(\frac{\cos^2 20° + \cos^2 70°}{\sin^2 25° + \sin^2 65°}\right) - \tan 45° + \tan 13° \tan 23° \tan 30° \tan 67° \tan 77°$

   (*ii*) $\frac{\sin^2 22° + \sin^2 68°}{\cos^2 22° + \cos^2 68°} + \sin^2 63° + \cos 63° \sin 27°$

3. If $\frac{4}{3}(\sec^2 59° - \cot^2 31°) - \frac{2}{3}\sin 90° + 3\tan^2 56° \tan^2 34° = \frac{x}{3}$, then find the value of x.

Prove the following (4 to 10) identities, where the angles involved are acute angles for which the trigonometric ratios are defined:

4. (*i*) $\frac{\cos A}{1 - \sin A} + \frac{\cos A}{1 + \sin A} = 2 \sec A$ (*ii*) $\frac{\cos A}{\operatorname{cosec} A + 1} + \frac{\cos A}{\operatorname{cosec} A - 1} = 2 \tan A$

(*iii*) $\frac{(\cos\theta - \sin\theta)(1 + \tan\theta)}{2\cos^2\theta - 1} = \sec\theta.$

5. (*i*) $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$

(*ii*) $\frac{\cot\theta}{\operatorname{cosec}\theta + 1} + \frac{\operatorname{cosec}\theta + 1}{\cot\theta} = 2\sec\theta.$

6. (*i*) $\sec^4 A(1 - \sin^4 A) - 2\tan^2 A = 1$

(*ii*) $\frac{1}{\sin A + \cos A + 1} + \frac{1}{\sin A + \cos A - 1} = \sec A + \operatorname{cosec} A.$

Hint

(*i*) $\sec^4 A (1 - \sin^4 A) = \sec^4 A - \tan^4 A$
$= (\sec^2 A + \tan^2 A)(\sec^2 A - \tan^2 A)$
$= ((1 + \tan^2 A) + \tan^2 A) \cdot 1$
$= 1 + 2\tan^2 A.$

7. (*i*) $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \sin\theta\cos\theta = 1$ (*ii*) $(\sec A - \tan A)^2 (1 + \sin A) = 1 - \sin A.$

8. (*i*) $\frac{\cos A}{1 - \tan A} - \frac{\sin^2 A}{\cos A - \sin A} = \sin A + \cos A$

(*ii*) $(\sec A - \operatorname{cosec} A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A \operatorname{cosec} A$

(*iii*) $\frac{\tan^2\theta}{\tan^2\theta - 1} + \frac{\operatorname{cosec}^2\theta}{\sec^2\theta - \operatorname{cosec}^2\theta} = \frac{1}{\sin^2\theta - \cos^2\theta}.$

9. $\frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} = \frac{1}{\sin^2 A - \cos^2 A} = \frac{2}{1 - 2\cos^2 A} = \frac{2\sec^2 A}{\tan^2 A - 1}.$

10. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0.$

11. If $\cot\theta + \cos\theta = m$ and $\cot\theta - \cos\theta = n$, then prove that $(m^2 - n^2)^2 = 16\,mn.$

12. When $0° < \theta < 90°$, solve the following equations:

(*i*) $2\cos^2\theta + \sin\theta - 2 = 0$ (*ii*) $3\cos\theta = 2\sin^2\theta$

(*iii*) $\sec^2\theta - 2\tan\theta = 0$ (*iv*) $\tan^2\theta = 3(\sec\theta - 1).$